

# CALABI–YAU COMPACTIFICATIONS OF TORIC LANDAU–GINZBURG MODELS FOR SMOOTH FANO THREEFOLDS

VICTOR PRZYJALKOWSKI

**ABSTRACT.** We prove that smooth Fano threefolds have toric Landau–Ginzburg models. More precise, we prove that their Landau–Ginzburg models, presented as Laurent polynomials, admit fiberwise compactifications to families of K3 surfaces. We also give an explicit construction of Landau–Ginzburg models for del Pezzo surfaces depending on divisors on them.

## 1. INTRODUCTION

The Mirror Symmetry conjecture, one of the deepest recent ideas in mathematics, relates varieties or certain families of varieties — so called Landau–Ginzburg models. It states that every variety has a partner whose symplectic properties correspond to algebraic ones for the original variety and, vice versa, whose algebraic properties correspond to symplectic ones for the original variety.

There are several levels of Mirror Symmetry conjecture. In this paper we study *Mirror Symmetry Conjecture of variations of Hodge structures*. It relates the (symplectic) Gromov–Witten invariants of Fano varieties (which count the expected numbers of (rational) curves lying on the varieties) with periods of dual algebraic families. Givental suggested dual Landau–Ginzburg models and proved Mirror Symmetry Conjecture of variations of Hodge structures for smooth toric varieties and Fano complete intersections therein, see [Gi97]. Another description of Mirror Symmetry for toric varieties treats it as a duality between toric varieties (or polytopes that define them), see, for instance, [Ba94]. From this point of view Laurent polynomials appear naturally as anticanonical sections on toric varieties.

Based on these considerations a notion of *toric Landau–Ginzburg model* was proposed in [Prz13]. A toric Landau–Ginzburg model for a smooth Fano variety  $X$  of dimension  $n$  and a chosen divisor on it is a Laurent polynomial  $f_X$  in  $n$  variables satisfying three conditions. The period condition relates periods of the family of fibers of the map  $f_X: (\mathbb{C}^*)^n \rightarrow \mathbb{C}$  to the generating series of one-pointed Gromov–Witten invariants of  $X$ . The second one, the Calabi–Yau condition, says that the fibers of the map  $f_X$  can be compactified to Calabi–Yau varieties. This condition is motivated by the so called Compactification Principle (see [Prz13, Principle 32]) stating that a fiberwise compactification of the family of the fibers of toric Landau–Ginzburg model is a Landau–Ginzburg model from the point of view of Homological Mirror Symmetry. Finally the toric condition is that there is a degeneration of  $X$  to a toric variety  $T$  whose fan polytope is the Newton polytope of  $f_X$ . This condition is motivated by the treatment of Mirror Symmetry for toric varieties as a duality between polytopes, and by the deformation invariance of the Gromov–Witten invariants.

The existence of toric Landau–Ginzburg models has been shown for Fano threefolds of Picard rank one and complete intersections (see [Prz13] and [ILP13]) in projective spaces. Aside from that only partial results are known. In particular, the existence of Laurent polynomials satisfying the period condition for all smooth Fano threefolds (and

---

This work is supported by the Russian Science Foundation under grant 14-50-00005.

zero divisors on them) with very ample anticanonical classes was shown in [CCG<sup>+</sup>], and the toric condition for them was checked in [IKKPS] and [DHKLP].

The main goal of the paper is to complete the proof of the existence of toric Landau–Ginzburg models for smooth Fano threefolds, in other words, to check the Calabi–Yau condition. There are 105 families of smooth Fano threefolds; among them 98 have the anticanonical classes which are very ample. We consider “good” toric degenerations of these, that is, the degenerations to Gorenstein toric varieties. Given a Fano threefold  $X$  and its Gorenstein toric degeneration  $T$  we denote by  $\Delta$  its fan polytope. It is reflexive, which means that its dual polytope  $\nabla$  is integral. Let us assume that a Laurent polynomial  $f_X$ , whose Newton polytope is  $\Delta$ , is of Minkowski type, i.e., its coefficients correspond to expansions of facets of  $\Delta$  to Minkowski sums of elementary polygons (Definition 4). Assume also that  $f_X$  satisfies the period condition for  $X$  and a zero divisor. (According to [CCG<sup>+</sup>] and [CCGK16], all 98 Fano threefolds with very ample anticanonical classes have such polynomials.) The family of fibers of the map given by  $f_X$  is a one-dimensional linear subsystem in the anticanonical linear system of a toric variety  $T^\vee$  whose fan polytope is  $\nabla$ . Since  $\Delta$  is three-dimensional and reflexive,  $T^\vee$  has a crepant resolution. One of the members of the family given by  $f_X$  is a boundary divisor of  $T^\vee$ . The base locus of the family is a union of smooth rational curves due to the special origin of coefficients of  $f_X$ . This enables one to resolve the base locus, and, cutting out the boundary divisor, to get a Calabi–Yau compactification (and to get a description of “the fiber over infinity” of the compactified Landau–Ginzburg model).

**Theorem 1** (Theorem 28). *Let  $f$  be a Minkowski Laurent polynomial. Then it admits a Calabi–Yau compactification.*

We also construct “by hand” Calabi–Yau compactifications for Laurent polynomials corresponding to seven Fano threefolds whose anticanonical classes are not very ample. More precisely, we compactify the families to families of (singular) quartics in  $\mathbb{P}^3$  or to hypersurfaces of bidegree  $(2, 3)$  in  $\mathbb{P}^1 \times \mathbb{P}^2$ , which have crepant resolutions. The following statement summarises the results discussed above.

**Corollary 2** (Corollary 32). *A pair of smooth Fano threefold  $X$  and a zero divisor on it has a toric Landau–Ginzburg model. Moreover, if  $-K_X$  is very ample, then any Minkowski Laurent polynomial satisfying the period condition for  $(X, 0)$  is a toric Landau–Ginzburg model.*

One can get Calabi–Yau compactifications for del Pezzo surfaces (of degrees greater than 2) in a similar way. In this case the base locus is just a set of points (possibly with multiplicities), and the resolution of the base locus is given just by a number of blow ups of these points. Landau–Ginzburg models for del Pezzo surfaces equipped with general divisors are constructed in [AKO06]; see also [UY13] for another description for arbitrary divisors. Our compactification procedure gives a precise method to write down a Laurent polynomial (and to describe singularities of its fibers) for *any* chosen divisor.

The paper is organized as follows. In Section 2 we recall some preliminaries and give the main notation and definitions. In particular, we define toric Landau–Ginzburg models and give the definition of Minkowski Laurent polynomials. In Section 3 we consider the two-dimensional case. We present an explicit way to write down a toric Landau–Ginzburg model for a del Pezzo surface with an arbitrary divisor. In Section 4 we study the case of Fano threefolds and zero divisors. We describe the structure of the base loci of the pencils we are interested in. Using this we prove the existence of Calabi–Yau compactifications for Minkowski Laurent polynomials (Theorem 28). Then we construct Calabi–Yau compactifications for the rest cases. We summarize our considerations in Corollary 32, where we state that smooth Fano threefolds have toric Landau–Ginzburg models. We also give some

remarks and questions on relation of reducible fibers of compactified Landau–Ginzburg models and their fibers over infinity.

**Notation and conventions.** All varieties are defined over the field  $\mathbb{C}$  of complex numbers. We use the standard notation for multidegrees: given an element  $a = (a_1, \dots, a_k) \in \mathbb{Z}^k$  and a multivariable  $x = (x_1, \dots, x_k)$ , we denote  $x_1^{a_1} \dots x_k^{a_k}$  by  $x^a$ . For a variety  $X$ , we denote  $\text{Pic}(X) \otimes \mathbb{C}$  by  $\text{Pic}(X)_{\mathbb{C}}$ . For a Laurent polynomial  $f \in \mathbb{C}[\mathbb{Z}^n]$  we denote its Newton polytope, i.e. a convex hull in  $\mathbb{Z}^n \otimes \mathbb{R}$  of non-zero monomials of  $f$ , by  $N(f)$ . Smooth del Pezzo surface of degree  $d$  (excluding the case of quadric surface) is denoted by  $S_d$ . We denote by  $X_{k-m}$  the smooth Fano variety of Picard rank  $k$  and number  $m$  on the lists in [IP99]. The missing variety of Picard rank 4 and degree 26 we denote by  $X_{4-2}$  and shift subscripts for varieties of the same Picard rank and greater degree. We denote by  $\mathbb{P}(a_0, \dots, a_n)$  The weighted projective space with weights  $a_0, \dots, a_n$ . The (weighted) projective space with coordinates  $x_0, \dots, x_n$  we denote by  $\mathbb{P}[x_0 : \dots : x_n]$ . The ring  $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$  is denoted by  $\mathbb{T}[x_1, \dots, x_n]$ .

**Acknowledgements.** The author is grateful to A. Corti, conversations with whom drastically improved the paper, and to V. Golyshev, A. Harder, N. Ilten, A. Kasprzyk, V. Lunts, D. Sakovics, and C. Shramov for helpful comments.

## 2. PRELIMINARIES

**2.1. Toric geometry.** Consider a toric variety  $T$ . A *fan* (or *spanning*) polytope  $F(T)$  is a convex hull of integral generators of fan’s rays for  $T$ . Let  $\Delta = F(T) \subset N_{\mathbb{R}} = \mathbb{Z}^n \otimes \mathbb{R}$ . Let

$$\nabla = \{x \mid \langle x, y \rangle \geq -1 \text{ for all } y \in \Delta\} \subset M_{\mathbb{R}} = N^{\vee} \otimes \mathbb{R}$$

be a dual polytope.

For an integral polytope  $\Delta$  we associate a (singular) toric Fano variety  $T_{\Delta}$  with it defined by a fan whose cones are cones over faces of  $\Delta$ . We also associate a (not uniquely defined) toric variety  $\tilde{T}_{\Delta}$  with  $F(\tilde{T}_{\Delta}) = \Delta$  such that for any toric variety  $T'$  with  $F(T') = \Delta$  and any morphism  $T' \rightarrow \tilde{T}_{\Delta}$  one has  $T' \simeq \tilde{T}_{\Delta}$ . In other words,  $\tilde{T}_{\Delta}$  is given by “maximal triangulation” of  $\Delta$ .

We denote  $T_{\nabla}$  by  $T^{\vee}$  and  $\tilde{T}_{\nabla}$  by  $\tilde{T}^{\vee}$ . In the rest of the paper we assume that  $\nabla$  is integral, in other words, that  $\Delta$  (or  $\nabla$ , or  $T$ , or  $T^{\vee}$ , or  $\tilde{T}^{\vee}$ ) is *reflexive*. In particular this means that integral points of both  $\Delta$  and  $\nabla$  are either the origin or lie on the boundary.

**Lemma 3.** *Let  $T$  be a threefold reflexive toric variety. Then  $\tilde{T}^{\vee}$  is smooth.*

*Proof.* Let  $C$  be a two-dimensional cone of the fan of  $\tilde{T}^{\vee}$ . It is a cone over an integral triangle  $R$  without strictly internal integral points that lie in the plane  $L = \{x \mid \langle x, y \rangle \geq -1\}$  for some  $y \in N$ . This means that is some basis  $e_1, e_2, e_3$  in  $M$  one gets  $L = \{a_1 e_1 + a_2 e_2 + e_3\}$ . Let  $P$  be a pyramid over  $R$  whose vertex is an origin. Then by Pick’s formula one has  $\text{vol } R = \frac{1}{2}$  and  $\text{vol } P = \frac{1}{6}$ , which means that vertices of  $R$  form basis in  $M$ , so  $\tilde{T}^{\vee}$  is smooth.  $\square$

Unfortunately, Lemma 3 does not hold for higher dimensions in general, because there are  $n$ -dimensional simplices whose only integral points are vertices, such that their volume is greater then  $\frac{1}{n!}$ . However it holds in some cases we are interested in.

**Definition 4** (see [CCGK16]). An integral polygon is called *of type  $A_n$* ,  $n \geq 0$ , if it is a triangle such that two its edges have integral length 1 and the rest one has integral length  $n$ . (In other words, its integral points are 3 vertices and  $n - 1$  ones lying on the same edge.) In particular,  $A_0$  is a segment of integral length 1.

An integral polygon  $P$  is called *Minkowski*, or *of Minkowski type*, if it is a Minkowski sum of some numbers of polytopes of type  $A_n$ , that is

$$P = \{p_1 + \dots + p_k \mid p_i \in P_i\}$$

for some polygons  $P_i$  of type  $A_{k_i}$ , and if the affine lattice generated by  $P \cap \mathbb{Z}^2$  is a sum of affine lattices generated by  $P_i \cap \mathbb{Z}^2$ . Such decomposition is called *admissible lattice Minkowski decomposition* and denoted by  $P = P_1 + \dots + P_k$ .

An integral three-dimensional polytope is called *Minkowski* if it is reflexive and if all its facets are Minkowski polygons.

*Remark 5* (cf. [CCGK16]). There are 105 smooth Fano threefolds (see [Is77], [Is78], and [MM82]). Among them 98 varieties have degenerations to Gorenstein toric varieties whose fan polytopes are Minkowski, and seven varieties, that are  $X_{1-1}$ ,  $X_{1-11}$ ,  $X_{2-1}$ ,  $X_{2-2}$ ,  $X_{2-3}$ ,  $X_{9-1}$ , and  $X_{10-1}$ , do not have Gorenstein toric degenerations.

Finally summarize some facts related to toric varieties and their anticanonical sections. One can see, say, [Da78] for details. It is more convenient to start from the toric variety  $T^\vee$  for the following.

- Fact 1. One can embed  $T^\vee$  to a projective space in the following way. Consider a set  $A \subset M$  of integral points in a polytope  $\Delta$  dual to  $F(T^\vee)$ . Consider a projective space  $\mathbb{P}$  whose coordinates  $x_i$  correspond to elements  $a_i$  of  $A$ . Associate a homogeneous equation  $\prod x_i^{\alpha_i} = \prod x_j^{\beta_j}$  with any homogenous relation  $\sum \alpha_i a_i = \sum \beta_j a_j$ ,  $\alpha_i, \beta_j \in \mathbb{Z}_+$ . The variety  $T^\vee$  is cut out in  $\mathbb{P}$  by equations associated to all homogenous relations on  $a_i$ .
- Fact 2. The anticanonical linear system of  $T^\vee$  is  $\mathcal{O}_{\mathbb{P}}(1)$ . In particular, it can be described as a linear system of Laurent polynomials whose Newton polytopes contain in  $\Delta$ .
- Fact 3. Toric strata of  $T^\vee$  of dimension  $k$  correspond to  $k$ -dimensional faces of  $\Delta$ . Denote by  $R_f$  an anticanonical section corresponding to a Laurent polynomial  $f \in \mathbb{C}[N]$  and by  $F_Q$  a strata corresponding to a face  $Q$  of  $\Delta$ . Denote by  $f|_Q$  a sum of those summands of  $f$  whose support lie in  $Q$ . Denote by  $\mathbb{P}_Q$  a projective space whose coordinates correspond to  $Q \cap N$ . (In particular,  $Q$  is cut out in  $\mathbb{P}_Q$  by homogenous relations on integral points of  $Q \cap N$ .) Then  $R_{Q,f} = R_f|_{F_Q} = \{f|_Q = 0\} \subset \mathbb{P}_Q$ .
- Fact 4. In particular,  $R_f$  does not pass through a toric point corresponding to a vertex of  $\Delta$  if and only if its coefficient at this point is non-zero. The constant Laurent polynomial corresponds to the boundary divisor of  $T^\vee$ .

**2.2. Laurent polynomials and toric Landau–Ginzburg models.** Consider a Laurent polynomial  $f$  and its Newton polytope  $\Delta = N(f)$ . Construct a toric variety  $T_\Delta$ . Different polynomials with the same Newton polytope give the same toric variety. However a choice of particular coefficients of a polynomial is important from Mirror Symmetry point of view.

We briefly remind a notion of toric Landau–Ginzburg model. More details see, say, in [Prz13].

Let  $X$  be a smooth Fano variety of dimension  $n$  and Picard number  $\rho$ . Let the number

$$\langle \tau_{a_1} \gamma_1, \dots, \tau_{a_k} \gamma_n \rangle_\beta, \quad a_i \in \mathbb{Z}_{\geq 0}, \quad \gamma_i \in H^*(X, \mathbb{C}), \quad \beta \in H_2(X, \mathbb{Z}),$$

be a  $k$ -pointed genus 0 Gromov–Witten invariant with descendants for  $X$ , see [Ma99, VI-2.1].

Choose a  $\mathbb{C}$ -divisors  $L$  and  $D$  on  $X$ . One can treat  $L$  as a direction (one-dimensional subtorus) on  $\mathbb{T} = \text{Spec } \mathbb{C}[H_2(X, \mathbb{Z})]$  and  $D$  as a point  $p_D$  on  $\mathbb{T}$ . Let  $\mathbf{1}$  be the fundamental class of  $X$ . Denote by  $R$  a set of classes of effective curves.

The series

$$I_0^{X,L,D}(t) = 1 + \sum_{\beta \in R, a \in \mathbb{Z}_{\geq 0}} \langle \tau_a \mathbf{1} \rangle_{\beta} \cdot e^{-\beta \cdot D} t^{\beta \cdot L}$$

is called a *constant term of  $I$ -series* (or a *constant term of Givental's  $J$ -series*) for  $(X, D)$ . It is a flat section for the restriction of the first Dubrovin's connection (see [Gi96]) on  $\mathbb{T}$  to a direction of  $D$  passing through  $p_D$ . The series

$$\tilde{I}_0^{X,L,D}(t) = 1 + \sum_{\beta \in R, a \in \mathbb{Z}_{\geq 0}} (\beta \cdot L)! \langle \tau_a \mathbf{1} \rangle_{\beta} \cdot e^{-\beta \cdot D} t^{\beta \cdot L}$$

is called a *constant term of regularized  $I$ -series* for  $X$ . It is a flat section for the restriction of the second Dubrovin's connection on  $\mathbb{T}$  to a direction of  $D$  passing through  $p_D$ .

Now we are interested in an *anticanonical direction*, so  $L = -K_X$ . We denote  $I_0^{X,-K_X,D}$  by  $I^{X,D}$  and  $\tilde{I}_0^{X,-K_X,D}$  by  $\tilde{I}^{X,D}$ . The latter series is the main object we associate to the pair  $(X, D)$ . Moreover, we often consider the case  $D = 0$ ; in this case we use notations  $I^X$  and  $\tilde{I}^X$  for  $I^{X,0}$  and  $\tilde{I}^{X,0}$  respectively.

Now consider the other side of Mirror Symmetry.

**Definition 6.** Let  $f$  be a Laurent polynomial in  $n$  variables  $x_1, \dots, x_n$ . The integral

$$I_f(t) = \frac{1}{(2\pi i)^n} \int_{|x_i|=\varepsilon_i} \frac{dx_1}{x_1} \wedge \dots \wedge \frac{dx_n}{x_n} \frac{1}{1-tf} = \frac{1}{(2\pi i)^n} \sum_{j=0}^{\infty} t^j \cdot \int_{|x_i|=\varepsilon_i} f^j \frac{dx_1}{x_1} \wedge \dots \wedge \frac{dx_n}{x_n} \in \mathbb{C}[[t]]$$

is called *the main period* for  $f$ , where  $\varepsilon_i$  are arbitrary positive real numbers.

*Remark 7.* Let  $\phi[g]$  be a constant term of a Laurent polynomial  $g$ . Then  $I_f(t) = \sum \phi[f^j] t^j$ .

The following theorem (which is a mathematical folklore, see [Prz08, Proposition 2.3] for the proof) justifies this definition.

**Theorem 8.** *Let  $f$  be a Laurent polynomial in  $n$  variables. Let  $P$  be a Picard–Fuchs differential operator for a pencil of hypersurfaces in a torus provided by  $f$ . Then one has  $P[I_f(t)] = 0$ .*

**Definition 9** (see [Prz13, §6]). A *toric Landau–Ginzburg model* for a pair of a smooth Fano variety  $X$  of dimension  $n$  and divisor  $D$  on it is a Laurent polynomial  $f \in \mathbb{T}[x_1, \dots, x_n]$  which satisfies the following.

**Period condition:** One has  $I_f(t) = \tilde{I}^{X,D}$ .

**Calabi–Yau condition:** There exists a relative compactification of a family

$$f: (\mathbb{C}^*)^n \rightarrow \mathbb{C}$$

whose total space is a (non-compact) smooth Calabi–Yau variety  $Y$ . Such compactification is called a *Calabi–Yau compactification*.

**Toric condition:** There is a degeneration  $X \rightsquigarrow T_X$  to a toric variety  $T_X$  such that  $F(T_X) = N(f)$ .

We consider Mirror Symmetry as a correspondence between Fano varieties and Laurent polynomials. That is, a strong version of Mirror Symmetry of variations of Hodge structures conjecture states the following.

**Conjecture 10** (see [Prz13, Conjecture 38]). *Any pair of a smooth Fano variety and a divisor on it has a toric Landau–Ginzburg model.*

In addition to threefold case discussed in the introduction, the existence of Laurent polynomials satisfying period condition for smooth toric varieties is shown in [Gi97], for complete intersections in Grassmannians is shown in [PSh14a], [PSh14b] [PSh15b], for



some complete intersections in some toric varieties is shown in [CKP14] and [DH15]. In [DH15] the toric condition for some complete intersections in toric varieties and partial flag manifolds is also checked.

In a lot of cases for “appropriate” polynomials satisfying period and toric conditions the Calabi–Yau condition is satisfied as well. However it is not easy to check this condition: there are no general enough approaches as for the two rest conditions are; usually one needs to check the Calabi–Yau condition “by hand”. The natural idea is to compactify the fibers of the map  $f: (\mathbb{C}^*)^n \rightarrow \mathbb{C}$  using the embedding  $(\mathbb{C}^*)^n \hookrightarrow T^\vee$ , where a toric variety  $T = T_\Delta$  corresponds to  $\Delta = N(f)$ . Indeed, the fibers compactify to anticanonical sections in  $T^\vee$ , and, since, have trivial canonical classes. However, first,  $T^\vee$  is usually singular, and, even if we resolve it (if it has a crepant resolution!), we can just conclude that its general anticanonical section is a smooth Calabi–Yau variety, but it is hard to say anything about the particular sections we need. Second, the pencil of anticanonical sections we are interested in has a base locus which we need to blow up to construct a Calabi–Yau compactification; and this blow up can be non-crepant.

Coefficients of polynomials that correspond to anticanonical divisors tend to have very symmetric coefficients, at least for the simplest toric degenerations. In this case the base loci are more simple and enable us to construct Calabi–Yau compactifications.

Consider a Laurent polynomial  $f$ . Remind that we can construct a Fano toric variety  $T = T_\Delta$ , where  $\Delta = N(f)$ , a dual toric Fano variety  $T^\vee = T_{\nabla}$ , where  $\nabla$  is a dual polytope for  $\Delta$ , and maximally triangulated toric variety  $\tilde{T}^\vee = \tilde{T}_{\nabla}$ . We make two assumptions for  $f$ . First, we assume that  $\nabla$  is integral, in other words, that  $\Delta$  is reflexive. In particular this means that integral points of both  $\Delta$  and  $\nabla$  are either the origin or lie on the boundary. The other assumption is related to special “symmetric” choice of coefficients of  $f$ .

**Definition 11** (see [CCGK16]). Let  $P \in \mathbb{Z}^2 \otimes \mathbb{R}$  be an integral polygon of type  $A_n$ . Let  $v_0, \dots, v_n$  be consecutive integral points on the edge of  $P$  of integral length  $n$  and let  $u$  be the rest integral point of  $P$ . Let  $x = (x_1, x_2)$  be a multivariable corresponding to an integral lattice  $\mathbb{Z}^2 \subset \mathbb{R}^2$ . Put

$$f_P = x^u + \sum \binom{n}{k} x^{v_k}.$$

(In particular one has  $f_P = x^u + x^{v_0}$  for  $n = 0$ .)

Let  $Q = Q_1 + \dots + Q_s$  be an admissible lattice Minkowski decomposition of an integral polygon  $Q \subset \mathbb{R}^2$ . Put

$$f_{Q_1, \dots, Q_s} = f_{Q_1} \cdot \dots \cdot f_{Q_s}.$$

A Laurent polynomial  $f \in \mathbb{T}[x_1, x_2, x_3]$  is called *Minkowski* if  $N(f)$  is Minkowski and for any facet  $Q \subset N(f)$ , as for an integral plane polytope, there exist an admissible lattice Minkowski decomposition  $Q = Q_1 + \dots + Q_s$  such that  $f|_Q = f_{Q_1, \dots, Q_s}$ .

### 3. DEL PEZZO SURFACES

We start the section by reminding well known facts about del Pezzo surfaces. We refer, say, to [Do12] as to one of huge amount of references on del Pezzo surfaces.

The initial definition of del Pezzo surface is the following one given by P. del Pezzo himself.

**Definition 12** ([dP87]). A *del Pezzo surface* is a non-degenerate irreducible linear normal (that is it is not a projection of degree  $d$  surface in  $\mathbb{P}^{d+1}$ ) surface in  $\mathbb{P}^d$  of degree  $d$  which is not a cone.

In modern words this means that a del Pezzo surface is an (anticanonically embedded) surface with ample anticanonical class and canonical (the same as du Val, simple surface,

Kleinian, or rational double point) singularities. (Classes of canonical and Gorenstein singularities for surfaces coincide.) So we use the following more general definition.

**Definition 13.** A *del Pezzo surface* is a complete surface with ample anticanonical divisor and canonical singularities. A *weak del Pezzo surface* is a complete surface with nef and big anticanonical divisor and canonical singularities.

*Remark 14.* Weak del Pezzo surfaces are (partial) minimal resolutions of singularities of del Pezzo surfaces. Exceptional divisors of the resolutions are  $(-2)$ -curves.

A *degree* of del Pezzo surface  $S$  is the number  $d = (-K_S)^2$ . One have  $1 \leq d \leq 9$ . If  $d > 2$ , then the anticanonical class of  $S$  is very ample and it gives an embedding  $S \hookrightarrow \mathbb{P}^d$ , so both definitions coincide. From now on we assume that  $d > 2$ .

Obviously, projecting a degree  $d$  surface in  $\mathbb{P}^d$  from a point on it one gets degree  $d - 1$  surface in  $\mathbb{P}^{d-1}$ . This projection is nothing but blow up of the center of the projection and blow down all lines passing through the point. (By adjunction formula these lines are  $(-2)$ -curves.) If we choose general (i. e. not lying on lines) centers of projections we get a classical description of smooth del Pezzo surface of degree  $d$  as a quadric surface (with  $d = 8$ ) or a blow up of  $\mathbb{P}^2$  in  $9 - d$  points. They degenerate to singular surfaces which are projections from non-general points (including infinitely close ones). Moreover, all del Pezzo surfaces of given degree lie in the same deformation space except for degree 8 when there are two components (one for a quadric surface and one for a blow up  $\mathbb{F}_1$  of  $\mathbb{P}^2$ ). General elements of the families are smooth, and all singular del Pezzo surfaces are degenerations of the smooth ones in these families. This description enables us to construct toric degenerations of del Pezzo surfaces. That is,  $\mathbb{P}^2$  is toric itself. Projecting from toric points one gets a (possibly singular) toric del Pezzo surface.

*Remark 15.* Del Pezzo surfaces of degree 1 or 2 also have toric degenerations. However their singularities are worse then canonical.

Let  $T_S$  be a Gorenstein toric degeneration of a del Pezzo surface  $S$  of degree  $d$ . Let  $\Delta = F(T_S) \subset N_{\mathbb{R}} = \mathbb{Z}^2 \otimes \mathbb{R}$  be a fan polytope of  $T_S$ . Let  $f$  be a Laurent polynomial such that  $N(f) = \Delta$ .

Our goal now is to describe in details a way to construct a Calabi–Yau compactification for  $f$ . More precise, we construct a commutative diagram

$$\begin{array}{ccccc} (\mathbb{C}^*)^2 & \hookrightarrow & Y & \hookrightarrow & Z \\ & \searrow f & \downarrow & & \downarrow \\ & & \mathbb{A}^1 & \hookrightarrow & \mathbb{P}^1, \end{array}$$

where  $Y$  and  $Z$  are smooth, fibers of maps  $Y \rightarrow \mathbb{A}^1$  and  $Z \rightarrow \mathbb{P}^1$  are compact, and  $-K_Z = f^{-1}(\infty)$ ; we denote all “vertical” maps in the diagram by  $f$  for simplicity.

The strategy is the following. First we consider a natural compactification of the pencil  $\{f = \lambda\}$  to an elliptic pencil in a toric del Pezzo surface  $T^{\vee}$ . Then we resolve singularities of  $T^{\vee}$  and get a pencil in smooth toric weak del Pezzo surface  $\tilde{T}^{\vee}$ . Finally we resolve basic locus of the pencil to get  $Z$ . We get  $Y$  cutting out strict transform of the boundary divisor of  $\tilde{T}^{\vee}$ . We keep track the geometry and cohomology of the objects under consideration constructing the Calabi–Yau compactification.

The polygon  $\Delta$  has integral vertices in  $N_{\mathbb{R}}$  and it has the origin as a unique strictly internal integral point. A dual polygon  $\nabla = \Delta^{\vee} \subset M = N^{\vee}$  has integral vertices as well. Combinatorially in dimension 2 this means that both  $\Delta$  and  $\nabla$  are integral polytopes with a unique strictly internal integral point, and geometrically this means that singularities of  $T$  and  $T^{\vee}$  are canonical.

*Remark 16.* The normalized volume of  $\nabla$  is given by

$$\text{vol } \nabla = |\text{integral points in } \nabla| - 1 = (-K_S)^2 = d.$$

It is easy to see that

$$|\text{integral points in } \Delta| + |\text{integral points in } \nabla| = 12.$$

In particular,  $\text{vol } \Delta = 12 - d$ .

*Compactification construction 17.* By Fact 2, anticanonical linear system for  $T^\vee$  can be described as a projectivisation of a linear space of Laurent polynomials whose Newton polytopes are contained in  $\nabla^\vee = \Delta$ . Thus the natural way to compactify the family is to do it using embedding  $(\mathbb{C}^*)^2 \hookrightarrow T^\vee$ . Fibers of the family are anticanonical divisors in this (possibly singular) toric variety. Two anticanonical sections intersect by  $(-K_{T^\vee})^2 = \text{vol } \Delta = 12 - d$  points (counted with multiplicities), so the compactification of the pencil in  $T^\vee$  has  $12 - d$  base points (possibly with multiplicities). The family  $\{f = \lambda, \lambda \in \mathbb{C}\}$  is generated by its general member and a divisor corresponding to a constant Laurent polynomial, i.e. to the boundary divisor of  $T^\vee$ . Let us mention that the torus invariant points of  $T$  do not lie in the base locus of the family by Fact 4.

Let  $\tilde{T}^\vee \rightarrow T^\vee$  be a minimal resolution of singularities of  $T^\vee$ . Pull back the pencil under consideration. We get an elliptic pencil with  $12 - d$  base points (with multiplicities), which are smooth points of one of members of the family, that is the boundary divisor  $D$  of the toric surface  $\tilde{T}^\vee$ ; this divisor is a wheel of  $d$  smooth rational curves. Blow up these base points and get an elliptic surface  $Z$ . Let  $E_1, \dots, E_{12-d}$  be the exceptional curves of the blow up  $\pi: Z \rightarrow \tilde{T}^\vee$ ; in particular,  $Z$  is not toric. Denote strict transform of  $D$  by  $D$  for simplicity. Then one has

$$-K_Z = \pi^*(-K_{\tilde{T}^\vee}) - \sum E_i = D + \sum E_i - \sum E_i = D.$$

Thus the anticanonical class  $-K_Z$  contains  $D$  and consists of fibers of  $Z$ . This, in particular, means that an open variety  $Y = Z \setminus D$  is a Calabi–Yau compactification of the pencil provided by  $f$ . This variety has  $e > 0$  sections, where  $e$  is a number of base points of the pencil in  $\tilde{T}$  counted *without* multiplicities.

Summarizing, we obtain a smooth family of elliptic curves  $f: Z \rightarrow \mathbb{P}^1$  with a wheel  $D$  of  $d$  smooth rational curves over  $\infty$ .

*Remark 18.* Let the polynomial  $f$  be general among ones with the same Newton polytope. Then singular fibers of  $Z \rightarrow \mathbb{P}^1$  are either curves with a single node or a wheel of  $d$  lines over  $\infty$ . By Noether formula one has

$$12\chi(\mathcal{O}_Z) = (-K_Z)^2 + e(Z) = e(Z),$$

where  $e(Z)$  is a topological Euler characteristic. Thus singular fibers for  $Z \rightarrow \mathbb{P}^1$  are  $d$  curves with one node and a wheel of  $d$  curves over  $\infty$ . This description is given in [AKO06].

*Remark 19.* One can compactify all Landau–Ginzburg models for all del Pezzo surfaces of degree at least three simultaneously. That is, all reflexive polygons are contained in the biggest polygon  $B$ , that has vertices  $(2, -1)$ ,  $(-1, 2)$ ,  $(-1, -1)$ . Thus fibers of all Landau–Ginzburg models can be simultaneously compactified to (possibly singular) anticanonical curves on  $T_{B^\vee} = \mathbb{P}^2$ . Blow up the base locus to construct a base points free family. However in this case a general member of the family can pass through toric points as it can happen that  $N(f) \subsetneq B$ . This means that some of exceptional divisors of the minimal resolution are extra curves in a wheel over infinity.

In other words, consider a triangle of lines and an elliptic curve on  $\mathbb{P}^2$ . A general member of the pencil given by  $f$  is an elliptic curve on  $\mathbb{P}^2$ . The total space of the Calabi–Yau compactification (together with a fiber over infinity) is a blow up of nine



intersection points (counted with multiplicities) of the elliptic curve and the triangle of lines. Exceptional divisors for points lying over vertices of the triangle are components of the wheel over infinity for the Calabi–Yau compactification; the others are either sections of the pencil or components of fibers over finite points.

Now describe toric Landau–Ginzburg models for del Pezzo surfaces and (toric) weak del Pezzo surfaces. That is, for a del Pezzo surface  $S$ , its Gorenstein toric degeneration  $T$  with fan polytope  $\Delta$ , its crepant resolution  $\tilde{T}$  with the same fan polytope, and a divisor  $D \in \text{Pic}(S)_{\mathbb{C}} \cong \text{Pic}(\tilde{T})_{\mathbb{C}}$ , we construct two Laurent polynomials  $f_{S,D}$  and  $f_{\tilde{T},D}$ , that are toric Landau–Ginzburg models for  $S$  and  $\tilde{T}$  correspondingly, by induction.

Let  $S \cong \mathbb{P}^1 \times \mathbb{P}^1$  be a quadric surface and let  $D_S$  be an  $(a, b)$ -divisor on it. Let  $T_1 = S$  and let  $T_2$  be a quadratic cone;  $T_1$  and  $T_2$  are the only Gorenstein toric degenerations of  $S$ . One has  $T_2 = \mathbb{F}_2$ , so let  $D_{\mathbb{F}_2} = \alpha s + \beta f$ , where  $s$  is a section of  $\mathbb{F}_2$ , so that  $s^2 = -2$ , and  $f$  is a fiber of the map  $\mathbb{F}_2 \rightarrow \mathbb{P}^1$ . Define

$$f_{S,D_S} = f_{\tilde{T}_1,D_S} = x + \frac{e^{-a}}{x} + y + \frac{e^{-b}}{y}$$

for the first toric degeneration and

$$f_{S,D_S} = y + e^{-a} \frac{1}{xy} + (e^{-a} + e^{-b}) \frac{1}{y} + e^{-b} \frac{x}{y}, \quad f_{\tilde{T}_2,D_{\mathbb{F}_2}} = y + \frac{e^{-\beta}}{xy} + \frac{e^{-\alpha}}{y} + \frac{x}{y}$$

for the second one.

Now assume that  $S$  is a blow up of  $\mathbb{P}^2$ . First let  $S = T = \tilde{T} = \mathbb{P}^2$ , let  $l$  be a class of a line on  $S$ , and let  $D = a_0 l$ . Then up to a toric change of variables one has

$$f_{(\mathbb{P}^2,D)} = x + y + \frac{e^{-a_0}}{xy}.$$

Now let  $S'$  be a blow up of  $\mathbb{P}^2$  in  $k$  points with exceptional divisors  $e_1, \dots, e_k$ , let  $S$  be a blow up of  $S'$  in a point, and let  $e_{k+1}$  be an exceptional divisor for the blow up. We identify divisors on  $S'$  and their strict transforms on  $S$ , so  $\text{Pic}(S') = \text{Pic}(\tilde{T}') = \mathbb{Z}l + \mathbb{Z}e_1 + \dots + \mathbb{Z}e_k$  and  $\text{Pic}(S) = \mathbb{Z}l + \mathbb{Z}e_1 + \dots + \mathbb{Z}e_k + \mathbb{Z}e_{k+1}$ . Let  $D' = a_0 l + a_1 e_1 + \dots + a_k e_k \in \text{Pic}(S')_{\mathbb{C}}$  and  $D = D' + a_{k+1} e_{k+1} \in \text{Pic}(S)_{\mathbb{C}}$ . First describe the polynomial  $f_{\tilde{T},D}$ . Combinatorially  $\Delta = F(\tilde{T})$  is obtained from a polytope  $\Delta' = F(\tilde{T}')$  by adding one integral point  $K$ , that corresponds to the exceptional divisor  $e_{k+1}$ , and taking a convex hull. Let  $L, R$  be boundary points of  $\Delta$  neighbor to  $K$ , left and right with respect to clockwise order, let  $c_L$  and  $c_R$  be coefficients in  $f_{\tilde{T}',D'}$  at monomials corresponding to  $L$  and  $R$ . Let  $M \in \mathbb{T}[x, y]$  be a monomial corresponding to  $K$ . Then from Givental's description of Landau–Ginzburg models for toric varieties (see [Gi97]) one gets

$$f_{\tilde{T},D} = f_{\tilde{T}',D'} + c_L c_R e^{-a_{k+1}} M.$$

The polynomial  $f_{S,D}$  differs from  $f_{\tilde{T},D}$  by coefficients at non-vertex boundary points. For any boundary point  $K \subset \Delta$  define *marking*  $m_K$  as a coefficient of  $f_{\tilde{T},D}$  at  $K$ . Consider a facet of  $\Delta$  and let  $K_0, \dots, K_r$  be integral points in clockwise order of this facet. Then coefficient of  $f_{S,D}$  at  $K_i$  is a coefficient at  $s^i$  in the polynomial

$$m_{K_0} \left( 1 + \frac{m_{K_1}}{m_{K_0}} s \right) \cdot \dots \cdot \left( 1 + \frac{m_{K_r}}{m_{K_{r-1}}} s \right).$$

**Example 20.** Let  $S = S_7$ , let  $\Delta$  be the polytope with vertices  $(0, 1)$ ,  $(1, -1)$ ,  $(-1, -1)$ ,  $(-1, 0)$ , and let  $D = a_0 l + a_1 e_1 + a_2 e_2$ . Then

$$f_{\tilde{T},D} = x + y + e^{-a_0} \frac{1}{xy} + e^{-(a_0+a_1)} \frac{1}{y} + e^{-(a_0+a_1+a_2)} \frac{x}{y},$$

$$f_{S,D} = x + y + e^{-a_0} \frac{1}{xy} + (e^{-(a_0+a_1)} + e^{-(a_0+a_2)}) \frac{1}{y} + e^{-(a_0+a_1+a_2)} \frac{x}{y}.$$

*Remark 21.* One has  $\text{Pic}(S) \cong \text{Pic}(\tilde{T})$ . That is, if  $S$  is not a quadric, both  $\text{Pic}(S)$  and  $\text{Pic}(\tilde{T})$  are obtained by a sequence of blow ups in points (the only difference is that the points for  $\tilde{T}$  can lie on exceptional divisors of previous blow ups). Thus in both cases Picard groups are generated by a class of a line on  $\mathbb{P}^2$  and exceptional divisors  $e_1, \dots, e_k$ . However an image of  $e_i$  under the map of Picard groups given by the degeneration of  $S$  to  $\tilde{T}$  can be not equal to  $e_i$  itself but to some linear combination of the exceptional divisors. In other words these bases do not agree with the degeneration map.

*Remark 22.* The spaces parameterizing Landau–Ginzburg models for  $S$  and for  $\tilde{T}$  are the same — they are the spaces of Laurent polynomials with Newton polytope  $\Delta$  modulo toric rescaling. Thus any Laurent polynomial correspond to different elements of  $\text{Pic}(S)_{\mathbb{C}} \cong \text{Pic}(\tilde{T})_{\mathbb{C}}$ . This gives a map  $\text{Pic}(S)_{\mathbb{C}} \rightarrow \text{Pic}(\tilde{T})_{\mathbb{C}}$ . However this map is transcendental because of exponential nature of the parametrization.

**Proposition 23.** *The Laurent polynomial  $f_{S,D}$  is a toric Landau–Ginzburg model for  $(S, D)$ .*

*Proof.* It is well known that del Pezzo surfaces have a description as either smooth toric varieties or complete intersections in smooth toric varieties. This enables one to compute a series  $I^S$  and, since,  $\tilde{I}^{S,D}$  following [Gi97]. Using this it is straightforward to check that the period condition for  $f_{S,D}$  holds. The Calabi–Yau condition holds by Compactification construction 17. Finally the toric condition holds by construction.  $\square$

**Proposition 24.** *Consider two different Gorenstein toric degenerations  $T_1$  and  $T_2$  of a del Pezzo surface  $S$ . Let  $\Delta_1 = F(T_1)$  and  $\Delta_2 = F(T_2)$ . Consider families of Calabi–Yau compactifications of Laurent polynomials with Newton polytopes  $\Delta_1$  and  $\Delta_2$ . Then there is a birational isomorphism of these families. In other words, there is a birational isomorphism between linear spaces of Laurent polynomials with supports in  $\Delta_1$  and  $\Delta_2$  modulo toric change of variables that preserves Calabi–Yau compactifications.*

*Proof.* One can check that polytopes  $\Delta_1$  and  $\Delta_2$  differ by (a sequence of) mutations (see, say, [ACGK12]). These mutations agree with fiberwise birational isomorphisms of toric Landau–Ginzburg models modulo change of basis in  $H^2(S, \mathbb{Z})$  by the construction. The statement follows from the fact that birational elliptic curves are isomorphic.  $\square$

*Remark 25.* Let  $D = 0$ . Then the polynomial  $f_{S,0}$  has coefficients 1 at vertices and  $\binom{n}{k}$  at  $k$ -th integral point of an edge of length  $n$ . In other words,  $f_{S,0}$  is binomial.

#### 4. MINKOWSKI TORIC LANDAU–GINZBURG MODELS

**Lemma 26.** *Let  $f$  be a Minkowski Laurent polynomial. Then for any face  $Q$  of  $\Delta$  the curve  $R_{Q,f}$  is a union of transversally intersecting smooth rational curves (possibly with multiplicities).*

*Proof.* Let  $Q$  be of type  $A_n$ ,  $n > 0$ . In appropriate basis  $Q$  has vertices  $u = (0, 1)$ ,  $v_0 = (0, 0)$ ,  $v_a = (a, 0)$ , and integral points  $v_i = (i, 0)$ . Let  $x, x_0, \dots, x_a$  be coordinates corresponding to  $u, v_0, \dots, v_a$ . Then  $F_Q$  is given by relations  $x_i x_j = x_r x_s$ ,  $i + j = r + s$ , in  $\mathbb{P}[x : x_0 : \dots : x_a]$ . This means that  $F_Q = v_a(\mathbb{P}(1, 1, a))$  is an image of  $a$ -th Veronese map of  $\mathbb{P}(1, 1, a)$ . Let  $y_0, y_1, y_2$  be coordinates on  $\mathbb{P}(1, 1, a)$ , where a weight of  $y_2$  is  $a$ . One has  $R_{Q,f} = \{\sum x_i \binom{n}{i} + x = 0\} \cap F_Q \subset \mathbb{P}^Q$ . Hence  $R_{Q,f} = \{(y_0 + y_1)^a + y_2 = 0\} \subset \mathbb{P}(1, 1, a)$ , so  $R_{Q,f}$  projects isomorphically to  $\mathbb{P}^1$  under projection of  $\mathbb{P}(1, 1, a)$  on the first two coordinates. Thus  $R_{Q,f}$  is a smooth rational curve with multiplicity one.

Now let  $Q = Q_1 + \dots + Q_k$  be a Minkowski decomposition, where  $Q_i$  is of type  $A_{k_i}$ , such that  $f|_Q = f_{Q_1} \cdot \dots \cdot f_{Q_k}$ . Then there is a Segre embedding  $\varphi: F_Q \rightarrow F_{Q_1} \times \dots \times F_{Q_k}$ . Let  $\pi_i$  be a projection  $F_{Q_1} \times \dots \times F_{Q_k} \rightarrow F_{Q_i}$ . Any map  $\pi_i \varphi$  is either birational or a  $\mathbb{P}^1$ -bundle (if  $Q_i$  is of type  $A_0$ ). Moreover, if it is birational, then each component of  $R_{Q,f}$  maps isomorphically. Thus  $R_{Q,f}$  is a union of curves isomorphic to  $R_{Q_i,f}$  for  $k_i > 0$  and lines (fibers of  $\pi_i \varphi_i$ ) for  $k_i = 0$  (some of which may coincide). This gives the assertion of the lemma.  $\square$

**Proposition 27.** *Let  $f: W \rightarrow \mathbb{P}^1$  be a one-dimensional family such that  $Y$  is a smooth threefold,  $-K_W = D$  for  $D = f^{-1}(\infty)$ ,  $D$  is reduced and a base locus  $B \subset D$  is a union of smooth curves (possibly with multiplicities) such that for any two components  $D_1, D_2$  of  $D$  one has  $D_1 \cap D_2 \not\subset B$ . Then there is a smooth threefold  $Z$  such that the diagram*

$$\begin{array}{ccc} Z & \xrightarrow{\varphi} & Y \\ & \searrow & \downarrow f \\ & & \mathbb{P}^1 \end{array}$$

*is commutative and  $-K_Z = (\varphi f)^{-1}(\infty)$ .*

*Proof (cf. Compactification construction 17).* Let  $\pi: W' \rightarrow W$  be a blow up of one component  $C$  of  $B$  on  $W$ . Since  $\pi$  is a blow up of a smooth curve on a smooth variety,  $W'$  is smooth. Let  $E$  be an exceptional divisor of the blow up. Let  $D'$  be a proper transform of  $D$ . Since multiplicity of  $C$  in  $D$  is 1, one gets

$$-K_{W'} = \pi^*(-K_W) - E = D' + E - E = D'.$$

Moreover, a base locus of the family on  $W'$  is the same as  $B$ , or as  $B \setminus C$ , or as a union of  $B$  or  $B \setminus C$  and a curve  $C' = D \cap E$  which is isomorphic to  $C$ . (There are no isolated base points as the base locus is an intersection of two divisors on a smooth variety.) Thus all conditions of the proposition hold for  $W'$ . Since  $(W, f)$  is a canonical pair, the base locus  $B$  can be resolved in finite number of blow ups. This gives the required resolution  $\varphi$ .  $\square$

**Theorem 28.** *Let  $f$  be a Minkowski Laurent polynomial. Then it admits a Calabi–Yau compactification.*

*Proof.* Remind that the Newton polytope  $\Delta$  of  $f$  is three-dimensional and reflexive, and the (singular Fano) toric variety whose fan polytope is  $\nabla = \Delta^\vee$  is denoted by  $T^\vee$ . The family of fibers of the map given by  $f$  is a family  $\{f = \lambda\}$ ,  $\lambda \in \mathbb{C}$ . Members of this family have natural compactifications to anticanonical sections in  $T^\vee$ . This family (more precise, its compactification to a family  $\{\lambda_0 f = \lambda_1\}$  over  $\mathbb{P}[\lambda_0 : \lambda_1]$ ) is generated by a general member and the member that corresponds to the constant Laurent polynomial. The latter is nothing but the boundary divisor  $D$  of  $T^\vee$ . Denote the obtained family on  $T^\vee$  by  $f: Z_{T^\vee} \rightarrow \mathbb{P}^1$  (we use the same notation  $f$  for the Laurent polynomial, the corresponding family, and resolutions of this family for simplicity). By Corollary 26, the base locus of  $Z_{T^\vee}$  is a union of smooth (rational) curves (possibly with multiplicities). By Lemma 3, the variety  $\tilde{T}^\vee$  is a crepant resolution of  $T^\vee$ . Since  $f$  is non-degenerate, that is its coefficients at vertices of  $\Delta$  are non-zero, the base locus does not contain any torus invariant strata of  $T^\vee$  — it does not contain torus invariant points. Thus we get a family  $f: Z_{\tilde{T}^\vee} \rightarrow \mathbb{P}^1$ , whose total space is smooth and a base locus is a union of transversally intersecting smooth curves (possibly with multiplicities) again. By Proposition 27, there is a resolution  $f: Z \rightarrow \mathbb{P}^1$  of the base locus of  $Z_{\tilde{T}^\vee}$  such that  $Z$  is smooth and  $-K_Z = f^{-1}(\infty)$ . Thus  $Y = Z \setminus f^{-1}(\infty)$  is the required Calabi–Yau compactification.  $\square$

*Remark 29.* The construction of Calabi–Yau compactification is not canonical: it depends on an order of blow ups of base locus components. However all of them are isomorphic in codimension one.

There are 105 smooth Fano threefolds. By Remark 5 and [CCGK16], there are 98 ones among them that have degenerations to toric varieties whose fan polytopes coincide with Newton polytopes of Minkowski Laurent polynomials satisfying period condition (for zero divisors). Since they have toric Landau–Ginzburg models by Proposition 27. Two other varieties,  $X_{1-1}$  and  $X_{1-11}$ , have toric Landau–Ginzburg models by [Prz13]. The following proposition, in a spirit of [Prz13], proves the existence of Calabi–Yau compactifications for some degenerations of other threefolds.

**Proposition 30.** *Fano threefolds  $X_{2-1}$ ,  $X_{2-2}$ ,  $X_{2-3}$ ,  $X_{9-1}$ , and  $X_{10-1}$  have toric Landau–Ginzburg models.*

*Proof.* The Fano variety  $X_{2-1}$  is a hypersurface section of type  $(1, 1)$  in  $\mathbb{P}^1 \times X_{1-11}$  in an anticanonical embedding; in other words, it is a complete intersection of hypersurfaces of type  $(1, 1)$  and  $(0, 6)$  in  $\mathbb{P}^1 \times \mathbb{P}(1, 1, 1, 2, 3)$ . The Fano variety  $X_{2-2}$  is a hypersurface in a certain toric variety, see [CCGK16]. The Fano variety  $X_{2-3}$  is a hyperplane section of type  $(1, 1)$  in  $\mathbb{P}^1 \times X_{1-12}$  in an anticanonical embedding; in other words, it is a complete intersection of hypersurfaces of type  $(1, 1)$  and  $(0, 4)$  in  $\mathbb{P}^1 \times \mathbb{P}(1, 1, 1, 1, 2)$ . Finally one has  $X_{9-1} = \mathbb{P}^1 \times S_2$  and  $X_{10-1} = \mathbb{P}^1 \times S_1$ .

For a variety  $X_{i-j}$  construct its Givental’s type Landau–Ginzburg models, see [Gi97], for a compact explanation see [PSh14a]. Then present it by Laurent polynomial  $f_{i-j}$ , see, say, [Prz11] and [Prz13]. By [CCG<sup>+</sup>] it satisfies the period condition, and by [IKKPS] and [DHKLP] it satisfies the toric condition. In a spirit of [Prz13] compactify the family given by  $f_{i-j}$  to a family of (singular) anticanonical hypersurfaces in  $\mathbb{P}^1 \times \mathbb{P}^2$  or  $\mathbb{P}^3$  and then crepantly resolve singularities of a total space of the family. Consider these cases one by one.

Givental’s Landau–Ginzburg model for  $X_{2-1}$  is a complete intersection

$$\begin{cases} u + v_0 = 0, \\ v_1 + v_2 + v_3 = 0 \end{cases}$$

in  $\text{Spec } \mathbb{T}[u, v_0, v_1, v_2, v_3]$  with a function

$$u + \frac{1}{u} + v_0 + v_1 + v_2 + v_3 + \frac{1}{v_1 v_2^2 v_3^3}.$$

After birational change of variables (see [Prz11])

$$v_1 = \frac{x}{x + y + 1}, \quad v_2 = \frac{y}{x + y + 1}, \quad v_3 = \frac{1}{x + y + 1}, \quad u = \frac{z}{z + 1}, \quad v_0 = \frac{1}{z + 1}$$

one, up to an additive shift, gets a function

$$f_{2-1} = \frac{(x + y + 1)^6(z + 1)}{xy^2} + \frac{1}{z}$$

on a torus  $\text{Spec } \mathbb{T}[x, y, z]$ .

Consider a family  $\{f_{2-1} = \lambda\}$ ,  $\lambda \in \mathbb{C}$ . Make a birational change of variables (cf. the proof of Theorem 18 from [Prz13])

$$x = \frac{1}{b_1} - \frac{1}{b_1^2 b_2} - 1, \quad y = \frac{1}{b_1^2 b_2}, \quad z = \frac{1}{a_1} - 1$$

and multiply the obtained expression by a denominator. We see that the family is birational to

$$\{(1 - a_1)b_2^3 = ((1 - a_1)\lambda - a_1) a_1(b_1 b_2 - b_1^2 b_2 - 1)\} \subset \mathbb{A}[a_1, b_1, b_2].$$

Now this family can be compactified to a family of hypersurfaces of bidegree  $(1, 1)$  in  $\mathbb{P}^1 \times \mathbb{P}^2$  using the embedding  $\mathbb{T}[a_1, b_1, b_2] \hookrightarrow \mathbb{P}[a_0 : a_1] \times \mathbb{P}[b_0 : b_1 : b_2]$ . The (non-compact) total space of the family has trivial canonical class and its singularities are a union of rational curves which are du Val along a line in general points and (possibly) ordinary double points. Blow up any of these curves. We get singularities of the similar type again. After several crepant blow ups one approaches to a threefold with just ordinary double points; these points admit algebraic small resolution. This resolution completes the construction of Calabi–Yau compactification. Note that the total space  $(\mathbb{C}^*)^3$  of the initial family is embedded to the resolution.

In the similar way one gets Calabi–Yau compactifications for the other varieties. One has

$$f_{2-2} = \frac{(x + y + z + 1)^2}{x} + \frac{(x + y + z + 1)^4}{yz}.$$

A change of variables

$$x = ab, \quad y = bc, \quad z = c - ab - bc - 1$$

applied to a family  $f_{2-2}$  and a multiplication on denominators give a family of quartics

$$ac^3 = (c - ab - bc - 1)(\lambda ab - c^2).$$

The embedding  $\text{Spec } \mathbb{T}[a, b, c] \hookrightarrow \mathbb{P}[a : b : c : d]$  gives a compactification to a family of singular quartics over  $\mathbb{A}^1$ .

One has

$$f_{2-3} = \frac{(x + y + 1)^4(z + 1)}{xyz} + z + 1.$$

A change of variables

$$x = ac, \quad y = a - ac - 1, \quad z = \frac{b}{c} - 1$$

applied to a family  $f_{2-3}$  and a multiplication on denominators give a family

$$a^3b = (\lambda c - b)(b - c)(a - ac - 1).$$

The embedding  $\text{Spec } \mathbb{T}[a, b, c] \hookrightarrow \mathbb{P}[a : b : c : d]$  gives a compactification to a family of singular quartics over  $\mathbb{A}^1$ .

One has

$$f_{9-1} = x + \frac{1}{x} + \frac{(y + z + 1)^4}{yz}.$$

A change of variables

$$x = \frac{c}{b}, \quad y = ac, \quad z = a - ac - 1$$

applied to a family  $f_{9-1}$  and a multiplication on denominators give a family

$$a^3b = (\lambda bc - b^2 - c^2)(a - ac - 1).$$

The embedding  $\text{Spec } \mathbb{T}[a, b, c] \hookrightarrow \mathbb{P}[a : b : c : d]$  gives a compactification to a family of singular quartics over  $\mathbb{A}^1$ .

One has

$$f_{10-1} = \frac{(x + y + 1)^6}{xy^2} + z + \frac{1}{z}.$$

A change of variables

$$x = \frac{1}{b_1} - \frac{1}{b_1^2 b_2} - 1, \quad y = \frac{1}{b_1^2 b_2}, \quad z = a_1$$

applied to a family  $f_{10-1}$  and a multiplication on denominators give a family

$$a_1 b_2^3 = (\lambda a_1 - a_1^2 - 1)(b_1 b_2 - b_1^2 b_2 - 1).$$



The embedding  $\text{Spec } \mathbb{T}[a_1, b_1, b_2] \hookrightarrow \mathbb{P}[a_0 : a_1] \times \mathbb{P}[b_0 : b_1 : b_2]$  gives a compactification to a family of singular hypersurfaces of bidegree  $(1, 1)$  in  $\mathbb{P}^1 \times \mathbb{P}^2$  over  $\mathbb{A}^1$ .

In all cases total spaces of all these families have crepant resolutions.  $\square$

In some cases the Calabi–Yau compactification can be constructed in another way, using multipotential technique (see, say, [KP12]) and elliptic fibrations.

**Proposition 31** (A. Harder). *The polynomial  $f_{10-1}$  satisfies the Calabi–Yau condition.*

*Proof.* Consider a surface  $B = \mathbb{A}[w] \times \mathbb{P}[s_0 : s_1]$ . Compactify a family given by  $f_{10-1}$  to a family of elliptic curves over  $B$  so that the projection onto  $\mathbb{A}^1$  gives (the partial compactification of) the initial family. This Weierstrass fibration can be given by the equation

$$Y^2 = X^3 + f_2 X + f_3$$

with

$$f_2 = -\frac{1}{3} (2w^2 s_1^2 - 3ws_0 s_1 + s_0^2 + s_1^2)^4,$$

$$f_3 = \frac{2}{27} (2w^2 s_1^2 - 3ws_0 s_1 - 864ws_1^2 + s_0^2 + 864s_0 s_1 + s_1^2) (2w^2 s_1^2 - 3ws_0 s_1 + s_0^2 + s_1^2)^5.$$

This fibration is singular and has degeneracy locus over  $B$  given by the equation

$$s_1 (ws_1 - s_0) (2w^2 s_1^2 - 3ws_0 s_1 - 432ws_1^2 + s_0^2 + 432s_0 s_1 + s_1^2) \cdot (2w^2 s_1^2 - 3ws_0 s_1 + s_0^2 + s_1^2)^{10} = 0.$$

Each component of this singular locus is a smooth curve in  $B$ . The singularities in the total space of this fibration are in the fibers over the curve given by

$$2w^2 s_1^2 - 3ws_0 s_1 + s_0^2 + s_1^2 = 0.$$

Above this curve, we get a curve of du Val singularities of type  $E_8$ . These singularities can be resolved by blowing up 8 times. This gives a smooth variety  $Y$  which is relatively compact fibered over  $\mathbb{A}^1$ . To see that this resolution is actually a Calabi–Yau variety, one can use the canonical bundle formula ([Mi83, p. 132]). The equation is basically  $K_Y = f^*(K_B + L)$ , where  $f$  is a map  $Y \rightarrow B$ , and  $L$  is a divisor on the base of the fibration, which is in this case is the pullback from  $\mathbb{P}^1$  to  $B$  of section of  $\mathcal{O}_{\mathbb{P}^1}(2)$ . Therefore,  $K_Y$  is the pullback of the trivial bundle on  $B$ , hence is itself trivial. Thus  $Y$  is a Calabi–Yau compactification of the family given by  $f_{10-1}$ .  $\square$

Summarizing [Prz13], Theorem 28, and Proposition 30 one gets the following assertion.

**Corollary 32.** *A pair of smooth Fano threefold  $X$  and a zero divisor on it has a toric Landau–Ginzburg model. Moreover, if  $-K_X$  is very ample, then any Minkowski Laurent polynomial satisfying the period condition for  $(X, 0)$  is a toric Landau–Ginzburg model.*

**Problem 33.** *Prove this for smooth Fano threefolds and any divisor. A description of Laurent polynomials for all Fano threefolds and any divisor is contained in [DHKLP].*

**Question 34.** *Is it true that the Calabi–Yau condition follows from the period and the toric ones? If not, what conditions should be put on a Laurent polynomial to hold the implication?*

Another advantage of the compactification procedure described in Theorem 28 is that this construction enables one to describe “fibers of compactified toric Landau–Ginzburg models over infinity”. These fibers play an important role, say, for computing Landau–Ginzburg Hodge numbers, see [KKP14] and [LP16] for detailed studying of the del Pezzo case. We summarize these considerations in the following assertion.

**Corollary 35.** *Let  $f$  be a Minkowski Laurent polynomial. Let  $\tilde{T}^\vee$  be a (smooth) maximally triangulated toric variety such that  $F(\tilde{T}^\vee) = N(f)$ , and let  $D$  be a boundary divisor of  $\tilde{T}^\vee$ . There is a compactification of the family  $f: (\mathbb{C}^*)^3 \rightarrow \mathbb{C}$  to a family  $f: Z \rightarrow \mathbb{P}^1$  such that  $Z$  is smooth and  $-K_Z = f^{-1}(\infty) = D$ . In particular,  $D$  consists of  $\left(-K_{T_{N(f)}}\right)^3$  components combinatorially given by a triangulation of a sphere.*

*Remark 36.* The description of fibers of Landau–Ginzburg models over infinity as boundary divisors fits well to Mirror Symmetry considerations from the point of view of [CKP13] and [IKKPS]. In these papers Fano varieties and their Landau–Ginzburg models are connected, via their toric degenerations, by elementary transformations called *basic links*. From our point of view they are given by elementary subtriangulations of a sphere of boundary divisors.

General fibers of compactified toric Landau–Ginzburg models for Fano threefolds are smooth K3 surfaces. However some of them can be singular and even reducible. Our considerations give almost no information about them; however singular fibers of Landau–Ginzburg models are of special interest — they contain information about derived categories of singularities. There is a lack of examples of computing these categories. More computable invariant is a number of components of reducible fibers.

**Conjecture 37** ([PSh15a, Conjecture 1.1], see also [GKR12]). *Let  $X$  be a Fano variety of dimension  $n$ . Let  $f_X$  be its toric Landau–Ginzburg model corresponding to a zero divisor on  $X$ . Let  $k_{f_X}$  be a number of all components of all reducible fibers (without multiplicities) of a Calabi–Yau compactification for  $f_X$  minus the number of reducible fibers. One has*

$$h^{1,n-1}(X) = k_{f_X}.$$

This conjecture is proven for Fano threefolds of rank one (see [Prz13]) and for complete intersections (see [PSh15a]).

**Problem 38.** *Prove Conjecture 37 for all Fano threefolds.*

*Remark 39.* Most of Fano threefolds have “simple” toric degenerations, say, degenerations to toric varieties with cDV singularities (combinatorially this means that their fan polytopes have, except for the origin, integral points only on edges). In these particular cases one can keep track of the exceptional divisors appearing at the resolution procedure described in Proposition 27 and Theorem 28. That is, one can compute multiplicities of the base curves (each multiplicity greater than 1 gives exceptional divisors in fibers) and local behavior of their intersections. Then, in a way similar to [PSh15a, Resolution procedure 4.4], one can compute the required number of components.

## REFERENCES

- [ACGK12] M. Akhtar, T. Coates, S. Galkin, A. Kasprzyk, *Minkowski polynomials and mutations*, SIGMA Symmetry Integrability Geom. Methods Appl., 8:Paper 094, 17, 2012.
- [AKO06] D. Auroux, L. Katzarkov, D. Orlov, *Mirror symmetry for Del Pezzo surfaces: Vanishing cycles and coherent sheaves*, Inv. Math. 166, No. 3 (2006), 537–582.
- [Ba94] V. V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi–Yau hypersurfaces in toric varieties*, J. Algebraic Geom. 3 (1994), no. 3, 493–535.
- [CKP14] T. Coates, A. Kasprzyk, T. Prince, *Four-dimensional Fano toric complete intersections*, arXiv:1409.5030.
- [CKP13] I. Cheltsov, L. Katzarkov, V. Przyjalkowski, *Birational geometry via moduli spaces*, Birational geometry, rational curves, and arithmetic, 93–132, Springer, New York, 2013.
- [CCGK16] T. Coates, A. Corti, S. Galkin, A. Kasprzyk, *Quantum periods for 3-dimensional Fano manifolds*, Geom. Topol. 20 (2016), no. 1, 103–256.
- [CCG<sup>+</sup>] T. Coates, A. Corti, S. Galkin, V. Golyshev, A. Kasprzyk, *Fano varieties and extremal Laurent polynomials. A collaborative research blog*, <http://coates.ma.ic.ac.uk/fanosearch/>.

- [Da78] V. Danilov, *The geometry of toric varieties*, Russian Math. Surveys, 33:2 (1978), 97–154.
- [dP87] P. del Pezzo, *Sulle superficie dell' $n^{mo}$  ordine immerse nello spazio di dimensioni*, Rend. del circolo matematico di Palermo 1 (1): 241–271, 1887.
- [Do12] I. Dolgachev, *Classical algebraic geometry. A modern view*, Cambridge University Press, Cambridge, 2012.
- [DH15] C. Doran, A. Harder, *Toric Degenerations and the Laurent polynomials related to Givental's Landau–Ginzburg models*, Canad. J. Math. 68 (2016), no. 4, 784–815.
- [DHKLP] C. Doran, A. Harder, L. Katzarkov, J. Lewis, V. Przyjalkowski, *Modularity of Fano threefolds*, in preparation.
- [Gi96] A. Givental, *Equivariant Gromov–Witten invariants*, Int. Math. Res. Notices 13 (1996), 613–663.
- [Gi97] A. Givental, *A mirror theorem for toric complete intersections*, Topological field theory, primitive forms and related topics (Kyoto, 1996), 141–175, Progr. Math., 160, Birkhauser Boston, Boston, MA, 1998.
- [GKR12] M. Gross, L. Katzarkov, H. Ruddat, *Towards mirror symmetry for varieties of general type*, arXiv:1202.4042.
- [IKKPS] N. Ilten, A. Kasprzyk, L. Katzarkov, V. Przyjalkowski, D. Sakovics, *Projecting Fanos in the mirror*, in preparation.
- [ILP13] N. Ilten, J. Lewis, V. Przyjalkowski, *Toric Degenerations of Fano Threefolds Giving Weak Landau–Ginzburg Models*, Journal of Algebra 374 (2013), 104–121.
- [Is77] V. Iskovskikh, *Fano 3-folds I*, Math USSR, Izv. 11 (1977), 485–527.
- [Is78] V. Iskovskikh, *Fano 3-folds II*, Math USSR, Izv. 11 (1978), 469–506.
- [IP99] V. Iskovskikh, Yu. Prokhorov, *Fano varieties*, Encyclopaedia Math. Sci. 47 (1999) Springer, Berlin.
- [KKP14] L. Katzarkov, M. Kontsevich, T. Pantev, *Non-commutative Hodge structures of geometric origin and Tian–Todorov theorems*, arXiv:1409.5996.
- [KP12] L. Katzarkov, V. Przyjalkowski, *Landau–Ginzburg models — old and new*, Akbulut, Selman (ed.) et al., Proceedings of the 18th Gokova geometry–topology conference. Somerville, MA: International Press; Gokova: Gokova Geometry–Topology Conferences, 97–124 (2012).
- [LP16] V. Lunts, V. Przyjalkowski, *Landau–Ginzburg Hodge numbers for del Pezzo surfaces*, arXiv:1607.08880.
- [Ma99] Yu. Manin, *Frobenius manifolds, quantum cohomology, and moduli spaces*, Colloquium Publications. American Mathematical Society (AMS). 47. Providence, RI: American Mathematical Society (AMS) (1999).
- [Mi83] R. Miranda, *Smooth models for elliptic threefolds*, Birational geometry of degenerations, Summer Algebraic Geometry Semin., Harvard Univ. 1981, Prog. Math. 29, 85–133 (1983).
- [MM82] S. Mori, S. Mukai, *Classification of Fano 3-folds with  $B_2 \geq 2$* , Manuscripta Math., 36(2):147–162, 1981/82; erratum in Manuscripta Math. 110 (2003), no. 3, 407.
- [Prz08] V. Przyjalkowski, *On Landau–Ginzburg models for Fano varieties*, Comm. Num. Th. Phys., Vol. 1, No. 4, 713–728 (2008).
- [Prz11] V. Przyjalkowski, *Hori–Vafa mirror models for complete intersections in weighted projective spaces and weak Landau–Ginzburg models*, Cent. Eur. J. Math. 9, No. 5, 972–977 (2011).
- [Prz13] V. Przyjalkowski, *Weak Landau–Ginzburg models for smooth Fano threefolds*, Izv. Math. Vol., 77 No. 4 (2013), 135–160.
- [PSh14a] V. Przyjalkowski, C. Shramov, *Laurent phenomenon for Landau–Ginzburg models of complete intersections in Grassmannians of planes*, arXiv:1409.3729.
- [PSh14b] V. Przyjalkowski, C. Shramov, *On weak Landau–Ginzburg models for complete intersections in Grassmannians*, Russian Math. Surveys 69, No. 6, 1129–1131 (2014).
- [PSh15a] V. Przyjalkowski, C. Shramov, *On Hodge numbers of complete intersections and Landau–Ginzburg models*, Int. Math. Res. Not. IMRN, 2015:21 (2015), 11302–11332.
- [PSh15b] V. Przyjalkowski, C. Shramov, *Laurent phenomenon for Landau–Ginzburg models of complete intersections in Grassmannians*, Proc. Steklov Inst. Math., 290 (2015), 91–102.
- [UY13] K. Ueda, M. Yamazaki, *Homological mirror symmetry for toric orbifolds of toric del Pezzo surfaces*, J. Reine Angew. Math. 680 (2013), 1–22.

STEKLOV MATHEMATICAL INSTITUTE OF RUSSIAN ACADEMY OF SCIENCES, 8 GUBKINA STREET, MOSCOW 119991, RUSSIA

*E-mail address:* victorprz@mi.ras.ru, victorprz@gmail.com